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Real Parameter Margin Computation for Uncertain Structural Dynamic Systems

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This paper is concerned with the problem of computing real parameter margins for stabilized, structural dynamic systems with the masses, spring constants, and damping constants as uncertain parameters. Numerical algorithms for uncertain systems with multilinear parameters are investigated. It is shown that for a certain class of structural dynamic systems with small passive damping (e.g., flexible structures in space), the computational complexity of the problem can be avoided by modeling the system as a conservative plant, without loss of any practical significance.

I. Introduction

FOR lumped mass-spring-damper dynamical systems, the masses, spring constants, and damping constants appear multilinearly in the numerator and denominator of the plant transfer function.[1] The parameter margin computation for a stabilized mass-spring system using a frequency-sweeping approach (e.g., Ref. 2) based on the mapping theorem of Ref. 3 can become numerically sensitive because of possible discontinuities in some frequencies, as discussed in Ref. 1. It is analytically proved in Refs. 4 and 5 that the numerical sensitivity problem encountered in Ref. 1 is caused by actual discontinuities in frequency. A similar discontinuity issue for a frequency-sweeping approach to calculate the parameter margins is also discussed in Refs. 6 and 7.

It is shown in Ref. 5 that, for a stabilized, conservative system with multilinear uncertain parameters, one needs only to check for instability in the corner directions of the parameter space hypercube, at a finite number of critical frequencies. An alternative proof based on a geometrical interpretation of the mapping theorem was presented in Ref. 8.

Consequently, conservative plants have special properties that aid in parameter margin computations; i.e., the plant transfer function is real valued for all frequencies. This allows for the identification of compensator-dependent critical frequencies where the loop transfer function becomes real valued. If uncertain parameters appear multilinearly in the plant transfer function, real parameter margin computations can be further reduced to checking system stability at only a few critical frequencies and at only the corners of a parameter space hypercube. For nonconservative plants, however, some of the simplifications that are possible in real parameter margin computations for conservative plants are no longer valid. In that case, a more general algorithm is needed for computing real parameter margins (or real μ), which is of much current research interest.[9,10]

In practice, the damping constant is often the most uncertain parameter for structural dynamic systems. In that case, we may consider 1) a worst case with no damping, i.e., a conservative plant, 2) a case with fixed, nominal values of passive damping, or 3) a case in which the damping constant is considered as one of the uncertain parameters. In this paper, we investigate these cases to assess any practical significance of including the passive damping in determining the exact real parameter margin of stabilized, structural dynamic systems.

The remainder of this paper is organized as follows. In Sec. II, the concept of critical frequency and gain for stabilized, *conservative* plants with uncertain parameters is introduced, and a geometrical interpretation of the corner property of a conservative plant is presented. The multilinear property is exploited to simplify procedures for computing the ∞-norm real parameter margins. In Sec. III, for *nonconservative* plants with ℓ uncertain parameters, we employ a new algorithm proposed in Ref. 9 for real μ, by checking for instability in the corner directions of the $(\ell-2)$ dimensional *reduced parameter space hypercubes*. In Sec. IV, a generic example of stabilized, structural dynamic systems[1,4,5,11,12] is used to illustrate the concepts and algorithms.

II. Stabilized Conservative Plants

Critical Frequency and Gain

In this section, we present the concept of critical gain, as well as the critical frequency concept first introduced in Refs. 4 and 5, for stabilized conservative plants with uncertain parameters. No assumption is made about how the uncertain parameters appear in the coefficients of the plant transfer function or the closed-loop characteristic equation (linearly, multilinearly, etc.).

Consider a single-input/single-output (SISO) feedback control system with the closed-loop characteristic equation

$$1 + G(s,p)K(s) = 0 \tag{1}$$

where $G(s,p)$ is the transfer function of a conservative plant with an uncertain parameter vector $p = [p_1, \dots, p_\ell]^T$, $K(s)$ the compensator transfer function, and s the Laplace transform variable. Since the plant is conservative, $G(s,p)$ is a function of even powers of s and is a real number for every $s = j\omega$, where $j = \sqrt{-1}$; that is, $G(j\omega,p) = G(\omega^2,p)$. Thus, for a conservative plant, we have

$$1 + G(\omega^2,p)K(j\omega) = 0 \tag{2}$$

where $G(\omega^2,p)$ is real.

Let

$$K(j\omega) = \mathrm{Re}\left[K(j\omega)\right] + j\,\mathrm{Im}\left[K(j\omega)\right]$$

then Eq. (2) becomes

$$\left\{1 + G(\omega^2,p)\mathrm{Re}\left[K(j\omega)\right]\right\} + jG(\omega^2,p)\mathrm{Im}\left[K(j\omega)\right] = 0 \tag{3}$$

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Since the real and imaginary parts of Eq. (3) must be zero, we have the following two *critical instability constraints:*

$$G(\omega^2,p)\mathrm{Im}\left[K(j\omega)\right] = 0 \tag{4}$$

$$1 + G(\omega^2,p)\mathrm{Re}\left[K(j\omega)\right] = 0 \tag{5}$$

Note that a solution ω of $G(\omega^2,p)=0$ cannot be a solution of Eqs. (4) and (5). Consequently, Eq. (4) simplifies to

$$\mathrm{Im}\left[K(j\omega)\right] = 0$$

and it can be said that the closed-loop system with uncertain conservative plant becomes unstable only at frequencies that depend only on the compensator parameters, as first discovered in Refs. 4 and 5. Such a frequency, denoted by ω_c, is called the *critical frequency*, and the corresponding parameter vector p_c is called the *critical parameter vector*.

Solving for p_c from Eq. (5) for each ω_c is quite numerically complicated. However, it is interesting to notice that $G(\omega_c^2,p_c)$ can be expressed as

$$G(\omega_c^2,p_c) = \kappa_c G(\omega_c^2,\bar{p}) \tag{6}$$

where $\bar{p}$ is the nominal parameter vector and κ_c is a real scalar. Thus, Eq. (5) becomes

$$1 + \kappa_c G(\omega_c^2,\bar{p})\mathrm{Re}\left[K(j\omega_c)\right] = 0 \tag{7}$$

where κ_c is referred to as the *critical gain* that represents an overall gain change due to parameter variations.

Remark: A conventional root locus plot of the nominal closed-loop system vs overall loop gain may be used to identify the critical gains and frequencies where root loci cross the imaginary axis. The closed-loop system becomes unstable only at these critical frequencies, including $\omega=0$, for all possible parameter variations. Thus, the classical gain margin concept may still be used as a measure of the overall parameter robustness for a system whose uncertain parameters do not necessarily appear multilinearly. The smallest critical gain corresponds to the conventional *gain margin* of a SISO closed-loop system.

Critical Polynomial Equations

Since a SISO system composed of fixed compensation and uncertain conservative plant becomes unstable only at critical frequencies that depend only on the compensator parameters, the computation of the ∞-norm parameter margin can be performed as follows.

Let

$$K(j\omega) = \frac{\tilde{N}(j\omega)}{\tilde{D}(j\omega)} = \frac{\tilde{N}_r(\omega^2)+j\omega\tilde{N}_i(\omega^2)}{\tilde{D}_r(\omega^2)+j\omega\tilde{D}_i(\omega^2)} \tag{8}$$

and

$$G(j\omega,p) = G(\omega^2,p) = \frac{N(\omega^2,p)}{D(\omega^2,p)} \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (2), we obtain the *critical polynomial equations*

$$\omega\left[\tilde{N}_i(\omega^2)\tilde{D}_r(\omega^2) - \tilde{N}_r(\omega^2)\tilde{D}_i(\omega^2)\right] = 0 \tag{10}$$

$$D(\omega^2,p)\tilde{D}_r(\omega^2) + N(\omega^2,p)\tilde{N}_r(\omega^2) = 0 \tag{11}$$

For each critical frequency obtained by solving Eq. (10), we need to find the largest stable hypercube centered about $\bar{p}$ in parameter space. That is, the problem is to find p_c to minimize $\|\epsilon\|_\infty$, where $\epsilon=[\epsilon_1,\ldots,\epsilon_\ell]^T$ may be actual perturbations as in $p_i=\bar{p}_i+\epsilon_i$ or percentage variations as in $p_i=\bar{p}_i(1+\epsilon_i)$, subject to Eq. (11) for each critical frequency. Then the solution with the smallest magnitude becomes the ∞-norm parameter margin. The computation of such a parameter margin for general cases is quite numerically complicated. However, the ∞-norm parameter margin computation can be greatly simplified by making use of the multilinear property of conservative dynamical systems, as discussed in the following section.

Corner Directions in Parameter Space

As first shown in Ref. 5 for a stabilized conservative system with multilinear uncertain parameters, one needs only to check for instability in the corner directions of the parameter space hypercube, at a finite number of critical frequencies. An alternative geometrical proof of such an elegant corner (or vertex) property using the mapping theorem is presented here as follows.

For a SISO conservative system, the loop transfer function $G(s,p)K(s)$ becomes real valued at critical frequencies. When the mapping theorem of Ref. 3 is applied at critical frequencies to a SISO conservative system, with multilinear parameters, having fixed compensation and independent parameter perturbations, the convex hull of the $|G(j\omega,p)K(j\omega)|$ plane image of the parameter space hypercube collapses to a line segment on the real axis. Since the extreme points of the convex hull are defined by the vertices of the parameter space hypercube, the endpoints of this line segment correspond to one or more of these vertices. Therefore, it is sufficient to check system stability only in the corner directions of the parameter space hypercube for each critical frequency, and the largest stable hypercube touches the stability boundary on one of its corners.

The parameter margin is then defined by the parameter changes that cause an endpoint of the line segment to touch the critical stability point. In addition, the overall change in gain for the plant transfer function due to parameter changes is equal to the associated critical gain. When parameter margin computations for each corner of a parameter space hypercube and each critical frequency are completed, the parameter margin of smallest magnitude becomes the overall parameter margin for the system.

Eigenvalue Approach to Complex/Real Parameter Margin Computation

In this section, expanding on the basic concepts of the preceding sections, we develop an eigenvalue approach to computing complex or real parameter margins for stabilized conservative plants.

For a stabilized conservative plant with ℓ independently uncertain parameters that appear multilinearly, there are 2^ℓ hypercube corner directions that need to be checked in the ∞-norm real parameter margin computation. These corners correspond to the 2^ℓ possible combinations of parameter values, where the uncertain value may be an increase or a decrease in any particular parameter.

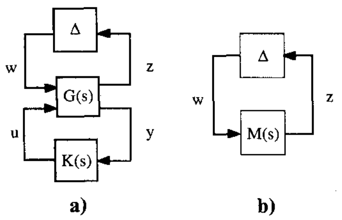

Fig. 1 Block diagram representations of a closed-loop control system with uncertain plant parameters.

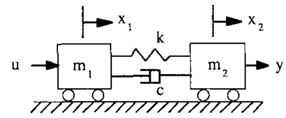

Fig. 2 Two-mass-spring-damper system with multilinear uncertain parameters.

Consider a closed-loop system illustrated by Fig. 1, where $G(s)$ is the nominal plant, $K(s)$ a stabilizing controller, Δ the structured uncertainty matrix, and $M(s) \in \mathcal{C}^{\ell \times \ell}$ the stable, nominal transfer function matrix from perturbation inputs w to perturbation outputs z. The closed-loop characteristic equation is then given by

$$\det\left[I - \Delta M(s)\right] = 0 \tag{12}$$

where $M(s)$ may contain input and output scaling factors; I is an identity matrix; and

$$\Delta := \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_\ell) \tag{13}$$

is the diagonal uncertainty matrix of independent parameter perturbations $\delta_i \in \mathcal{R}$.

Note that $\det[I - \Delta M(s)]$ is a polynomial of δ_i and is affine with respect to each δ_i and that the coefficients of the characteristic polynomial of the perturbed system are multilinear functions of δ_i.

Since only the corners of a parameter space hypercube are to be checked in ∞-norm parameter margin computations for a conservative plant with multilinear parameters, Δ can be expressed as

$$\Delta = \delta E \tag{14}$$

where $\delta \in [0, \infty)$ and

$$\mathcal{E} := \left[E : E = \operatorname{diag}(e_i),\ e_i = \pm 1 \forall i\right]$$

In this case, δ represents the size of the parameter space hypercube whereas the 2^ℓ possible sets of ± 1 in $\mathcal{E}$ define the 2^ℓ corner directions. The task is to find δ and the particular E matrix corresponding to the stable, parameter space hypercube and its particular corner that touches the stability boundary.

Let

$$\delta_c(\omega) := \min_{E \in \mathcal{E}} \left\{|\delta| : \det\left[I - \delta EM(j\omega)\right] = 0,\ \delta \in \mathcal{C}\right\} \tag{15}$$

$$\delta_c^* = \inf_\omega \delta_c(\omega) \tag{16}$$

where $\delta_c(\omega)$ and δ_c^* are called the *complex parameter robustness measure* and the *complex parameter margin*, respectively, for a conservative plant with complex multilinear parameter variations.

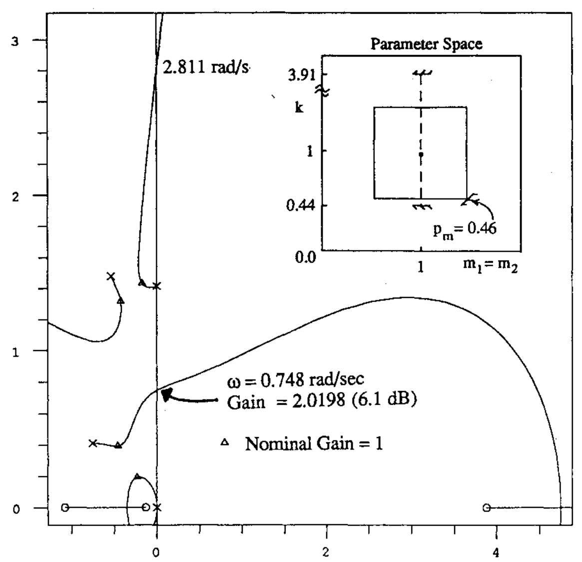

Fig. 3 Root locus of the nominal closed-loop system vs overall loop gain.[12]

Similarly, let

$$\delta_r(\omega) := \min_{E \in \mathcal{E}} \left\{|\delta| : \det\left[I - \delta EM(j\omega)\right] = 0, \quad \delta \in \mathcal{R}\right\} \tag{17}$$

$$\delta_r^* = \inf_\omega \delta_r(\omega) = \inf_{\omega_c} \delta_r(\omega) \tag{18}$$

where $\delta_r(\omega)$ and δ_r^* are called the *real parameter robustness measure* and the *real parameter margin,* respectively, for a conservative plant with real multilinear parameter variations, and ω_c are the *critical frequencies* defined in the previous section.

Remark: The *real parameter robustness measure* $\delta_r(\omega)$ of a conservative plant is discontinuous at each critical frequency, whereas the *complex parameter robustness measure* $\delta_c(\omega)$ of a conservative plant is continuous at all frequencies.

Since

$$\begin{aligned}\det\left[I - \delta EM(j\omega)\right] &= (\delta)^\ell \det\left[(1/\delta)I - EM(j\omega)\right]\\ &= (1/\lambda)^\ell \det\left[\lambda I - EM(j\omega)\right]\\ &= 0\end{aligned}$$

where λ can be interpreted as an eigenvalue of EM, the complex parameter margin can then be computed by simply finding the eigenvalues of $EM(j\omega)$ where $E \in \mathcal{E}$.

Thus, the complex parameter margin becomes

$$\delta_c^* = \inf_\omega \delta_c(\omega) = \inf_\omega \left| \left\{\max_{E \in \mathcal{E}} \lambda_{\max}\left[EM(j\omega)\right]\right\}^{-1} \right| \tag{19}$$

where $\lambda_{\max}[EM]$ denotes the eigenvalue of EM with the largest magnitude.

A further computational simplification is possible by making use of the sign of $\max \lambda_i [EM(j\omega)]$. By doing this, only one-half or $2^{\ell-1}$ of the corner directions need to be checked, and one of the diagonal elements in the E matrix may be fixed at $+1$ (or -1).

If we are interested only in the computation of a real parameter margin δ_r^*, then we can find it directly as follows:

$$\delta_r^* = \inf_{\omega_c} \delta_r(\omega) \tag{20}$$

where

$$\delta_r(\omega_c) = \delta_c(\omega_c) = \left| \left\{\max_{E \in \mathcal{E}} \lambda_{\max}\left[EM(j\omega)\right]\right\}^{-1} \right| \tag{21}$$

Note that $\lambda_{\max}[EM(j\omega_c)]$ is real.

III. Stabilized Nonconservative Plants

Some of the simplifications that are possible in real parameter margin computations for conservative plants are no longer valid for nonconservative plants. Consequently, a more general algorithm is needed for computing real parameter margins (or real μ), which is of much current research interest.[9,10]

A direct approach to such general cases is to define a parameter space hypercube, of dimension equal to the number of uncertain parameters and centered about the nominal parameter values, and then increase the size of the hypercube, always checking closed-loop stability for parameter values corresponding to points on the *surface* of the hypercube, until system instability occurs. Then the computation of the ∞-norm parameter margin corresponds to finding the largest stable hypercube in the parameter space. An obvious advantage of this method is that it is applicable for systems whose uncertain parameters do not necessarily appear multilinearly. The amount of computation increases dramatically as the number of parameters increases, however.

In this section, we investigate a new algorithm proposed by Dailey[9] for real μ. The algorithm is based on a conjecture that assumes that no more than two of the critical parameters do not have extreme values. While we check for instability in the corner directions of the ℓ-dimensional *parameter space hypercube* of a conservative plant, the new algorithm of Ref. 9 is basically checking for instability in the corner directions of the $(\ell-2)$ dimensional *reduced parameter space hypercubes* of a nonconservative plant.

Complex and Real Parameter Margins

Similar to the case of conservative plants, parameter robustness measures at each frequency can be defined as follows.

Let

$$\delta_c(\omega) := \min_{\Delta\in\mathfrak{D}_c}\left\{\bar{\sigma}(\Delta) : \det\left[I-\Delta M(j\omega)\right]=0\right\} \tag{22a}$$

$$\mathfrak{D}_c = \left\{\Delta : \Delta=\mathrm{diag}(\delta_i);\ \delta_i\in\mathfrak{C},\ i=1,\dots,\ell\right\} \tag{22b}$$

where $\delta_c(\omega)$ is called the *complex parameter robustness measure* for a nonconservative plant with multilinear parameters, which allows complex parameter variations.

Similarly, let

$$\delta_r(\omega) := \min_{\Delta\in\mathfrak{D}_r}\left\{\bar{\sigma}(\Delta) : \det\left[I-\Delta M(j\omega)\right]=0\right\} \tag{23a}$$

$$\mathfrak{D}_r = \left\{\Delta : \Delta=\mathrm{diag}(\delta_i);\ \delta_i\in\mathfrak{R},\ i=1,\dots,\ell\right\} \tag{23b}$$

where $\delta_r(\omega)$ is called the *real parameter robustness measure* for a nonconservative plant with multilinear parameters, which allows only real parameter variations.

The complex parameter margin δ_c^* and the real parameter margin δ_r^* are then defined as

$$\delta_c^* = \inf_{\omega}\delta_c(\omega) \tag{24a}$$

$$\delta_r^* = \inf_{\omega}\delta_r(\omega) \tag{24b}$$

Corner Directions in Reduced Parameter Subspaces

The algorithm to be discussed next was motivated by the work of Dailey[9] and was implemented as a MATLAB code in which the golden section search method is used. This algorithm only needs one dimensional linear search on the real parameter margin δ_r^* over [0,1] and solving a simple quadratic algebraic equation, without actually computing the determinant of $(I-\Delta M)$ and without checking if the edges of the convex hull intersect the origin.

The largest stable ℓ-dimensional hypercube is defined as

$$\mathfrak{A} := \left\{\alpha : \|\alpha\|_\infty \le \delta_r^*\right\}$$

where $\alpha=[\delta_1,\dots,\delta_\ell]^T$.

The $(\ell-2)$ dimensional *reduced parameter subspaces* of the ℓ-dimensional hypercube $\mathfrak{A}$ are then defined as

$$\mathfrak{B}_{ij} := \left\{\beta : \|\beta\|_\infty \le \delta_r^*\right\};\quad i\ne j\ \ (i,j=1,2,\dots,\ell)$$

where

$$\beta = [\delta_1,\delta_2,\dots,\delta_{i-1},\delta_{i+1},\dots,\delta_{j-1},\delta_{j+1},\dots,\delta_\ell]^T$$

which is formed by deleting the ith and jth elements from the ℓ-dimensional parameter vector α. There are total $\ell\,(\ell-1)/2$ reduced parameter subspaces (or reduced parameter space hypercubes) $\mathfrak{B}_{ij}$ for all possible (i,j) combinations.

The corners of the reduced parameter subspaces $\mathfrak{B}_{ij}$ ($i\ne j$; $i,j=1,2,\dots,\ell$) have the following property:

$$|\beta_k| = \delta_r^*,\qquad k=1,\dots,\ell-2$$

and all of the corners of the reduced parameter space hypercubes are referred to as the *reduced corners* of $\mathfrak{A}$.

In Ref. 9, the author conjectures that the worst case parameters occur at one of the reduced corners of $\mathfrak{A}$, and it can be easily shown that this conjecture is true for $\ell\le 3$.

We introduce the following function for the normalized parameter space:

$$\delta_{ij}(\omega) := \min_{\delta\in[0,1]}\left\{\delta : \det[I-\Delta_1 N]=0,\ \bar{\sigma}(\Delta_1)<\delta\right\} \tag{25}$$

where

$$\Delta_1 = \mathrm{diag}(\delta_i,\delta_j);\qquad \delta_i,\delta_j\in\mathfrak{R}$$

$$N = M_{11} + M_{12}(I-\Delta_2 M_{22})^{-1}\Delta_2 M_{21}$$

$$\Delta_2 = \delta E_2$$

$$\mathcal{E}_2 := \left[E_2 : E_2=\mathrm{diag}(e_k),\ e_k=\pm 1,\ k=1,\dots,\ell-2\right]$$

and M_{11}, M_{12}, M_{21}, and M_{22} are the corresponding parts in the partitioned matrix $(I-\Delta M)$ with $\Delta=\mathrm{diag}(\Delta_1,\Delta_2)$ such that for $\ell\ge 2$

$$(I-\Delta M) = \begin{bmatrix} I-\Delta_1 M_{11} & -\Delta_1 M_{12}\\ -\Delta_2 M_{21} & I-\Delta_2 M_{22}\end{bmatrix}$$

and M_{11} is a 2×2 matrix, M_{12} is a $2\times(\ell-2)$ matrix, M_{21} is an $(\ell-2)\times 2$ matrix, and M_{22} is an $(\ell-2)\times(\ell-2)$ matrix.

For $\ell\le 3$ (for all i and j)

$$\delta_r(\omega) = \min \delta_{ij}(\omega) \tag{26}$$

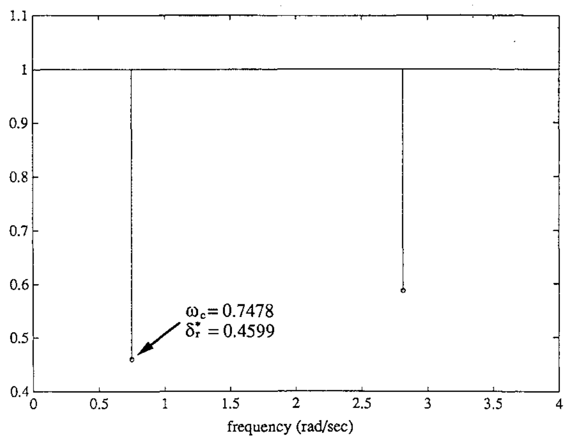

Fig. 4 Real parameter robustness measure $\delta_r(\omega)$ for a conservative case ($c=\bar{c}=0$).

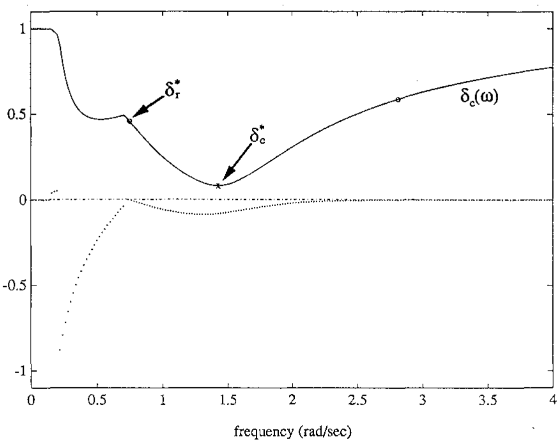

Fig. 5 Complex robustness measure $\delta_c(\omega)=1/\mu$ for a conservative case ($c=\bar{c}=0$).

and the real parameter margin simply becomes

$$\delta_r^* = \inf_{\omega} \delta_r(\omega) \tag{27}$$

For $\ell > 3$, we have the following inequality (for all i and j):

$$\delta_r(\omega) \leq \min \delta_{ij}(\omega) \tag{28}$$

IV. Two-Mass-Spring-Damper Example

In this section, a generic example of stabilized, structural dynamic systems is used to illustrate the concepts and algorithms presented in the preceding sections.

Structured Parameter Uncertainty Modeling

Consider a two-mass-spring-damper system shown in Fig. 2, which is a generic model of an uncertain dynamical system with a rigid-body mode and one vibration mode. A control force acts on body 1 and the position of body 2 is measured, resulting in a noncollocated control problem.

This system can be described as

$$m_1\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) = u$$

$$m_2\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = 0$$

$$y = x_2$$

where x_1 and x_2 are the positions of body 1 and body 2, respectively; $\dot{x}_1$ and $\dot{x}_2$ the velocities of body 1 and body 2, respectively; u the control input acting on body 1; y the measured output; m_1 and m_2 the masses of body 1 and body 2, respectively; k the spring stiffness coefficient; c the damping constant; and all parameters have the appropriate units and time is in units of seconds.

The transfer function from control input to measured output is

$$\frac{y(s)}{u(s)} = \frac{cs + k}{s^2\left[m_1 m_2 s^2 + c(m_1 + m_2)s + k(m_1 + m_2)\right]}$$

It can be seen that the uncertain parameters m_1, m_2, k, and c appear multilinearly in the numerator and denominator of the plant transfer function.

The uncertain parameters are then modeled as

$$m_1 = \bar{m}_1(1 + \delta_1)$$

$$m_2 = \bar{m}_2(1 + \delta_2)$$

$$k = \bar{k}(1 + \delta_3)$$

$$c = \bar{c}(1 + \delta_4)$$

where $\bar{m}_1$, $\bar{m}_2$, $\bar{k}$, and $\bar{c}$ are the nominal parameters, and δ_1, δ_2, δ_3, and δ_4 define percentage variations in each parameter.

The plant equations may now be written as

$$\bar{m}_1(1 + \delta_1)\ddot{x}_1 = \bar{c}(1 + \delta_4)(\dot{x}_2 - \dot{x}_1) + \bar{k}(1 + \delta_3)(x_2 - x_1) + u$$

$$\bar{m}_2(1 + \delta_2)\ddot{x}_2 = -\bar{c}(1 + \delta_4)(\dot{x}_2 - \dot{x}_1) - \bar{k}(1 + \delta_3)(x_2 - x_1)$$

$$y = x_2$$

After rearranging terms, we obtain

$$\bar{m}_1\ddot{x}_1 = \bar{c}(\dot{x}_2 - \dot{x}_1) + \bar{k}(x_2 - x_1) + u - w_1 + w_3 + w_4$$

$$\bar{m}_2\ddot{x}_2 = -\bar{c}(\dot{x}_2 - \dot{x}_1) - \bar{k}(x_2 - x_1) - w_2 - w_3 - w_4$$

where

$$w_1 = \delta_1 z_1, \qquad z_1 = \bar{m}_1\ddot{x}_l$$

$$w_2 = \delta_2 z_2, \qquad z_2 = \bar{m}_2\ddot{x}_2$$

$$w_3 = \delta_3 z_3, \qquad z_3 = \bar{k}(x_2 - x_1)$$

$$w_4 = \delta_4 z_4, \qquad z_4 = \bar{c}(\dot{x}_2 - \dot{x}_1)$$

and where w_i and z_i are referred to as the fictitious inputs and outputs, respectively.

This system is then described in state-space form with $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$ as

$$\dot{x} = Ax + B_1 w + B_2 u \tag{29a}$$

$$z = C_1 x + D_{11} w + D_{12} u \tag{29b}$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{29c}$$

$$w = \Delta z \tag{29d}$$

where

$$x = [x_1, x_2, x_3, x_4]^T$$

$$w = [w_1, w_2, w_3, w_4]^T$$

$$z = [z_1, z_2, z_3, z_4]^T$$

$$\Delta = \text{diag}(\delta_1, \delta_2, \delta_3, \delta_4) \tag{30}$$

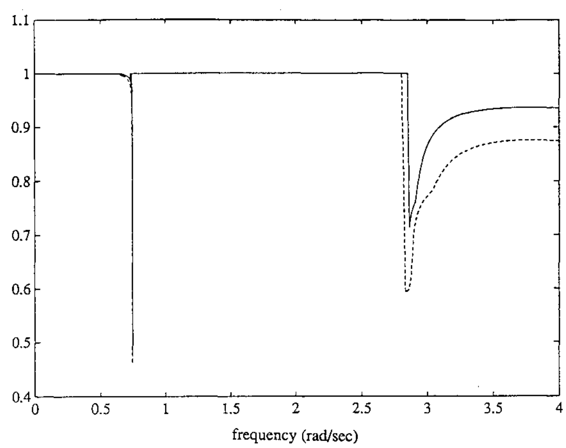

Fig. 6 Real parameter robustness measure for $\bar{c} = 0.0003$ [solid line for $c = \bar{c}$ and dashed line for $c = \bar{c}(1 + \delta_4)$].

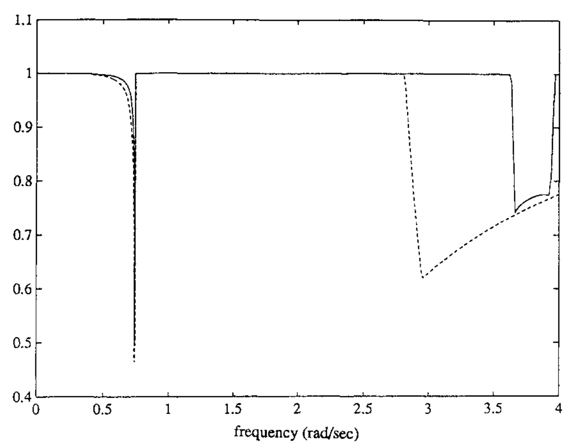

Fig. 7 Real parameter robustness measure for $\bar{c} = 0.00235$ [solid line for $c = \bar{c}$ and dashed line for $c = \bar{c}(1 + \delta_4)$].

and

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\bar{k}/\bar{m}_1 & \bar{k}/\bar{m}_1 & -\bar{c}/\bar{m}_1 & \bar{c}/\bar{m}_1 \\ \bar{k}/\bar{m}_2 & -\bar{k}/\bar{m}_2 & \bar{c}/\bar{m}_2 & -\bar{c}/\bar{m}_2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/\bar{m}_1 & 0 & 1/\bar{m}_1 & 1/\bar{m}_1 \\ 0 & -1/\bar{m}_2 & -1/\bar{m}_2 & -1/\bar{m}_2 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1/\bar{m}_1 & 0 \end{bmatrix}^T$$

$$C_1 = \begin{bmatrix} -\bar{k} & \bar{k} & -\bar{c} & \bar{c} \\ \bar{k} & -\bar{k} & \bar{c} & -\bar{c} \\ -\bar{k} & \bar{k} & 0 & 0 \\ 0 & 0 & -\bar{c} & \bar{c} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$D_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_{22} = 0$$

Conservative Case

Consider a case with zero damping. The two mass elements m_1 and m_2 and the spring constant k are assumed to be uncertain. The parameter space hypercube is three dimensional, which has $2^3 = 8$ corners of the cube centered about the assumed nominal parameter values:

$$[\bar{m}_1, \bar{m}_2, \bar{k}] = [1, 1, 1]$$

The nominal system has open-loop eigenvalues of $s = 0$, 0, $\pm\sqrt{2}j$ on the imaginary axis. The closed-loop stability robustness is to be analyzed for the following compensator of Ref. 12 given in transfer function form as

$$\frac{u(s)}{y(s)} = \frac{(-0.0728)\left[(-s/3.876)+1\right]\left[(s/1.075)+1\right]\left[(s/0.132)+1\right]}{\left[(s/0.86)^2 + 2(0.88/0.86)s + 1\right]\left[(s/1.57)^2 + 2(0.34/1.57)s + 1\right]}$$

or state-space form as

$$\dot{x}_c = A_c x_c + B_c y \tag{31a}$$

$$u = C_c x_c + D_c y \tag{31b}$$

where

$$A_c = \begin{bmatrix} 0.1250 & -0.2879 & 1.0587 & 0.0076 \\ -0.1116 & -0.5530 & -0.0524 & 0.9932 \\ -2.5747 & 1.8080 & -2.1485 & -0.2774 \\ 1.0069 & -1.1966 & 0.0042 & 0.0005 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0.1946 & 0.6791 & -0.0359 & 0.2013 \end{bmatrix}^T$$

$$C_c = \begin{bmatrix} -1.5717 & 0.7722 & -2.1450 & -0.2769 \end{bmatrix}$$

$$D_c = 0$$

and where x_c is the compensator state vector. The nominal closed-loop system has eigenvalues at $s = -0.2322 \pm 0.1919j$, $-0.4591 \pm 0.3936j$, $-0.4251 \pm 1.3177j$, and $-0.1717 \pm 1.4312j$.

Figure 3 shows a root locus of the nominal closed-loop system vs overall loop gain. The nominal system has a 6.1-dB gain margin and a 34-deg phase margin. Note that root loci cross the imaginary axis at $\omega = 0.747801$ rad/s ($\kappa_c = 2.0198$) and $\omega = 2.811543$ rad/s ($\kappa_c = 247.8$). As discussed in the preceding sections, this system becomes unstable at these two critical frequencies for all possible variations of m_1, m_2, and k.

The overall closed-loop system can be represented in state-space form as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{bmatrix} w \tag{32a}$$

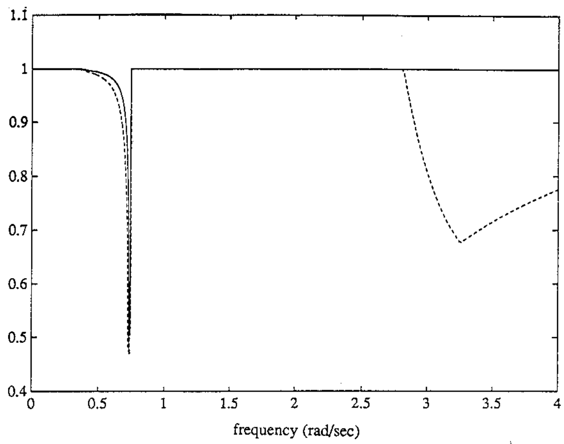

Fig. 8 Real parameter robustness measures for $\bar{c} = 0.006$ [solid line for $c = \bar{c}$ and dashed line for $c = \bar{c}(1 + \delta_4)$].

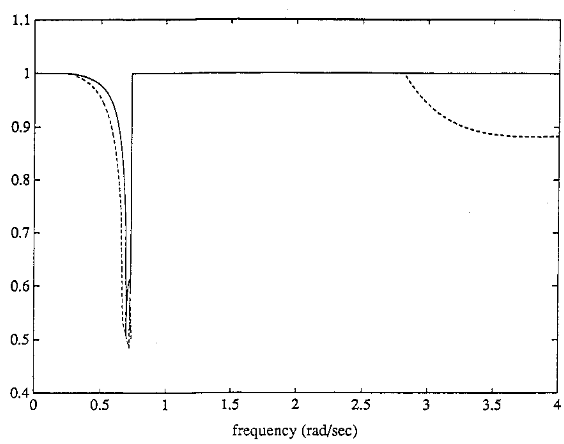

Fig. 9 Real parameter robustness measures $\bar{c} = 0.02$ [solid line for $c = \bar{c}$ and dashed line for $c = \bar{c}(1 + \delta_4)$].

$$z = \begin{bmatrix} C_1 + D_{12}D_cC_2 & D_{12}C_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + (D_{11} + D_{12}D_cD_{21})w \tag{32b}$$

or in transfer function form as

$$z = M(s)w \tag{33}$$

and

$$w = \Delta z$$

Note that Eqs. (32a) and (32b) can be further simplified when $D_c = 0$.

The critical frequencies can also be identified by solving Eq. (4) or Eq. (10), and $\delta_r(\omega_c)$ and the corresponding E can be found as

$$\omega_c = 0.747801; \quad \delta_r(\omega_c) = 0.459848; \quad E = \text{diag}(1, 1, -1)$$

$$\omega_c = 2.811534; \quad \delta_r(\omega_c) = 0.586918; \quad E = \text{diag}(-1, -1, 1)$$

Note that $\omega_c = 0$ is always a critical frequency, and the corresponding $\delta_r(0)$ can be found as one for this example. Thus, the ∞-norm real parameter margin for the system becomes

$$\delta_r^* = \inf_{\omega_c} \delta_r(\omega_c) = 0.459848$$

and the instability occurs at the critical corner at

$$[\delta_1, \delta_2, \delta_3] = [0.459848, 0.459848, -0.459848]$$

The corresponding critical parameter values are

$$[m_1, m_2, k] = [1.459848, 1.459848, 0.540152]$$

The real parameter robustness measure $\delta_r(\omega)$, which is discontinuous in frequency, is shown in Fig. 4. Since the system becomes unstable only at the two nonzero critical frequencies for all possible real parameter variations, $\delta_r(\omega) = 1$ at all other frequencies. This corresponds to the trivial case where some or all of the parameter values become zero.

Figure 5 shows a plot of $\delta_c(\omega) \equiv 1/\mu(\omega)$. The $\delta_c(\omega)$ values corresponding to real parameter variations [i.e., $\delta_r(\omega_c)$] are identified by circles. This figure clearly demonstrates that the parameter margin of 0.08, identified by a cross, predicted by the standard complex μ analysis is overly conservative, since the actual real parameter margin is 0.4599 for this example problem. The real parameter margin and the frequencies at which they occur may be identified in another way by monitoring

$$\text{Im}\left(\left\{\max_{E \in \mathcal{E}} \lambda_{\max}\left[EM(j\omega)\right]\right\}^{-1}\right)$$

which is indicated by a dotted line in Fig. 5. When this imaginary part is not zero, parameter margins are complex valued and are simply not physically possible.

Remark: For this conservative example problem, one of the corners with $m_1 = m_2$ always becomes the critical corner.

Nonconservative Case

In addition to the three uncertain parameters m_1, m_2, and k, the damping coefficient c is first assumed as a known nonzero constant.

To assess any practical significance of including passive damping, we consider four cases as summarized in Table 1. Since $\ell = 3$ for these cases, the use of the algorithm discussed in Sec. III gives the "exact" real parameter margins, and the results are summarized in Table 2. For all four cases, the critical instability occurs at one of the *corners* of the parameter space hypercube. From Table 2, we notice that the real parameter margins for cases 1, 2, and 3 with the passive damping ratio smaller than 0.5% are very close to the real parameter margin of 0.459848 of the zero-damping case.

We now consider cases in which the damping coefficient c is actually modeled as one of four uncertain parameters ($\ell = 4$). The nominal values of these parameters are taken as the same as given in Table 1. The real parameter margins have been found using the algorithm of Sec. III, and the results are summarized in Table 3. Rather surprisingly, for all four cases, the critical instability occurs at one of the corners of the parameter space hypercube. Again, real parameter margins are very close to the zero-damping case of 0.459848. In Figs. 6–9, the real parameter robustness measures vs frequency are shown for these four cases of Tables 2 and 3. The solid line plots are for the fixed damping cases and the dashed line plots for the uncertain damping cases.

Table 1 Nominal parameter values

	Case 1	Case 2	Case 3	Case 4
$\bar{c}$	0.0003	0.00235	0.006	0.02
$(\bar{\zeta})$	(0.00021)	(0.00166)	(0.00424)	(0.01414)
$\bar{m}_1$	1	1	1	1
$\bar{m}_2$	1	1	1	1
k	1	1	1	1

Table 2 Cases with fixed nominal damping

	Case 1	Case 2	Case 3	Case 4
$\bar{c}$	0.0003	0.00235	0.006	0.02
δ_r^*	0.460581	0.465581	0.474446	0.503351
ω_c	0.747022	0.741722	0.732363	0.740956
m_1	1.460581	1.465581	1.474446	0.496648
m_2	1.460581	1.465581	1.474446	0.496648
k	0.539419	0.534419	0.525554	0.496648

Table 3 Cases with uncertain damping

	Case 1	Case 2	Case 3	Case 4
$\bar{c}$	0.0003	0.00235	0.006	0.02
δ_r^*	0.460244	0.462929	0.467635	0.484840
ω_c	0.747380	0.744530	0.739548	0.721451
m_1	1.460244	1.462929	1.467635	1.484840
m_2	1.460244	1.462929	1.467635	1.484840
k	0.539756	0.537071	0.532365	0.515160
c	$0.539756\bar{c}$	$0.537071\bar{c}$	$0.532365\bar{c}$	$0.515160\bar{c}$

VI. Conclusions

The problem of finding the largest stable hypercube in the parameter space of uncertain structural dynamic systems has been investigated. The concept of the critical gains, critical frequencies, and critical parameters were introduced. An eigenvalue-based algorithm was then presented and shown to be very effective in computing the real parameter margin of a conservative plant. Another algorithm was also employed to assess any practical significance of including passive damping in computing the real parameter margin. It was shown that, for some practical cases with very small passive structural damping, the computational complexity of the problem can be simply avoided by modeling the system as a conservative plant, without significant loss in accuracy in predicting the real parameter margin.

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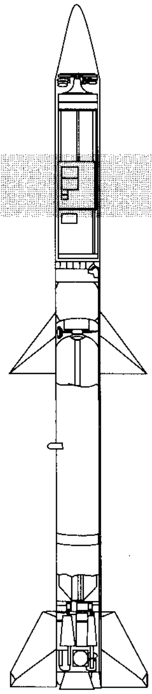

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